Topics in Applied Stochastic Processes Semester II 2020/21 Final Exam

Maximum Score: 100 points

Ground Rules:

- You may use the consolidated notes on the course website at: https://www.isibang.ac.in/∼athreya/Teaching/tas/tasnotes.pdf
- Please do not consult anyone else or any other resource: online sites, Homework solutions, books, notes, any other online notes etc.

Instructions:

- Please state and sign a declaration on page 1 of your submission stating that "I have read the ground rules and I have followed them".
- Please upload your answer as one pdf file on Moodle platform.
- You may use results done in class to solve quesitons (unless the question itself is the stated result) but please refer it appropriately w.r.t. notes or write the statement of the result.
- Please write as detailed and precise answers as possible.
- If you have any questions then please send me a private chat via zoom-contact.

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1. (20 points) Let $\{X_i\}_{i\geq 1}$ be a sequence of i.i.d. random variables such that $\mathbb{P}(X_i = 1) = \mathbb{P}(X_i = -1) = \frac{1}{2}$. Let $Z_1 = -2$ and
$$Z_{n+1} = Z_n(1 + X_n\left(\frac{3n+1}{n+1}\right))$$
Let $J = \min\{n \geq 2 : \text{sign}(Z_n) = \text{sign}(Z_{n-1})\}$ be a stopping time.
   (a) Show that $\{Z_n\}_{n\geq 1}$ is a martingale.
   (b) Using induction on n show that
$$\mathbb{P}(Z_n = -\frac{2^n}{n} \mid J > n) = 1 \qquad \text{and } \mathbb{P}(Z_n = -\frac{2^n(n-2)}{n^2 - n} \mid J = n) = 1$$
   (c) Show that $\mathbb{E} \mid Z_J \mid = \infty$ and $\lim_{n\to\infty} \mathbb{E}[Z_n \mid J > n]\mathbb{P}(J > n) = 0$

2. (25 points) Let $N \in \mathbb{N}$. Consider
$$\Omega_N = \{\omega = (\omega_1, \omega_2, \ldots, \omega_N) : \omega_i \in \{-1, +1\}$$
equipped with the uniform distribution, denoted by $\mathbb{P} \equiv \mathbb{P}_N$. For $1 \leq k \leq N$, let $X_K : \Omega_N \to \{-1, 1\}$ be given by $X_k(\omega) = \omega_k$ and for $1 \leq n \leq N$, let $S_n : \Omega_N \to \mathbb{Z}$ be given by $S_n(\omega) = \sum_{k=1}^{n} X_k(\omega)$ and $S_0 = 0$.
   (a) For $1 \leq n \leq N$, find the mode of S_n and the probability at the mode.
   (b) Let for $0 < k \leq N$, $a \in \mathbb{Z}, \mathbb{P}(S_k = a) > 0$. Prove that for $0 < k < m \leq N$,
$$\mathbb{P}(S_m = b \mid S_k = a) = \mathbb{P}(S_{m-k} = b - a)$$
   for $b \in \mathbb{Z}$.
   (c) Let $a \in \mathbb{N}$ and $\sigma_a = \min\{k \geq 1 : S_k = a\}$. Show that
$$\mathbb{P}(\sigma_a = n) = \frac{1}{2}[\mathbb{P}(S_{n-1} = a - 1) = -\mathbb{P}(S_{n-1} = a + 1)]$$
   (d) Let $\mathcal{A}_n$ be the events that are observable by time n. Find b_n so that $Y_n = S_n^2 - b_n$ is a martingale w.r.t $\mathcal{A}_n$.
   (e) Suppose X_i were all i.i.d. X. Let $m = \mathbb{E}[X]$ and $\sigma^2 = \text{Var}[X]$. Show that
$$\mathbb{P}(\max_{1\leq k\leq n} \mid S_k \mid \geq m\epsilon) \leq \frac{\sigma^2}{m\epsilon^2}$$

3. (15 points) Let the canopy tree $\mathcal{C}_2$ be the subgraph of $\mathbb{T}_2$.
   (a) Define $\mathcal{C}_2$
   (b) For $n \geq 1$, find the volume of the ball of radius n around ρ.
   (c) Show that there is only one infinite length self-avoiding path on $\mathcal{C}$.

4. (20 points) At time 0, an urn contains 1 black ball and 1 white ball. At each time $n \geq 1$, a ball is chosen from the urn and returned to the urn. At the same time, a new ball of the same colour as the chosen ball is added to the urn. Just after time n, there are $n + 2$ balls in the urn of which $B_n + 1$ are black. Let M_n be the proportion of balls in the urn that are black, at time n.
   (a) Show that M_n is a martingale.
   (b) Let T be the number of balls drawn until the first black ball appears. Show that T is a stopping time and find $\mathbb{E}[\frac{1}{T+2}]$.
   (c) Show that M_n converges to a random variable M as $n \to \infty$ with probability one and find the distribution of M.

5. (20 points) Rupali and Soha were studying the survival of family names in Wayanad, Kerala. Suppose each family has exactly 3 children but coin flips determine their sex. In Wayanad, only female children kept the family name. Suppose there was one female at time 0 with family name Charukesi. Compute the probability ρ that the family name Charukesi will die out.